THE BINOMIAL THEOREM

Imagine expanding $(a + b)^{100}$ using regular algebra. This chapter will show us a neat shortcut for solving this type of problem. The concepts involved in raising a binomial to a high power have applications in probability, from the flipping of coins to quality control in manufacturing light bulbs.

❑ *EXPANDING AND SEEING PATTERNS*

The goal of the Binomial Theorem is to expand powers of binomials, things like $(a + b)^{100}$, without actually multiplying anything out. Let's start with some expansions we already know how to do using brute-force algebra.

$(a+b)^0 = 1$ (anything [but 0] to the zero power is 1)

$(a+b)^1 = a + b$ (anything to the 1st power is itself)

$(a+b)^2 = a^2 + 2ab + b^2$ (just multiply $a + b$ by $a + b$)

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (multiply $a^2 + 2ab + b^2$ by $a + b$)

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ (etc.)

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ (etc.)

Homework

1. Verify the calculations above for each expansion:

   $(a+b)^2$ $(a+b)^3$ $(a+b)^4$ $(a+b)^5$

2. How many terms are there in each expansion?

   a. $(a+b)^0$ b. $(a+b)^1$ c. $(a+b)^2$

   d. $(a+b)^3$ e. $(a+b)^4$ f. $(a+b)^5$

3. a. How many terms are there in the expansion of $(a+b)^{100}$?

   b. How many terms are there in the expansion of $(a+b)^n$?

4. a. Notice the term $10a^3b^2$ in the expansion of $(a+b)^5$. What is the *sum* of the exponents on the a and the b?

   b. In the same expansion, the 5th term is $5ab^4$. What is the *sum* of the exponents on the a and the b?

5. a. See the first term in the expansion of $(a+b)^4$? It's a^4. What is the *sum* of the exponents on the a and the b? [Hint: Maybe there's no explicit b in the term a^4, but what power of b could be placed next to the a^4 so that it remains a^4? That is, $a^4b^{\textbf{?}} = a^4$.]

   b. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a+b)^4$?

6. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a+b)^n$?

7. If the expansion of $(a+b)^k$ has a term of the form $ca^{44}b^{33}$, where c is a constant, what is the value of k?

❑ OBSERVATIONS ON THE EXPANSIONS

1. The **number of terms** in each expansion is one more than the exponent on the binomial. For example, the expansion of $(a + b)^5$ has **6** terms, and the expansion of $(a + b)^n$ has $\boldsymbol{n + 1}$ terms.

2. In any given term, the **sum of the exponents** always equals the exponent on the binomial. For instance, looking at the expansion of $(a + b)^5$ from page 1:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

If we "patch up" the expansion of $(a + b)^5$ to include both the a and the b in every term, we can write

$$(a + b)^5 = a^5\boldsymbol{b^0} + 5a^4\boldsymbol{b^1} + 10a^3b^2 + 10a^2b^3 + 5\boldsymbol{a^1}b^4 + \boldsymbol{a^0}b^5$$

We notice that in any of the 6 terms, the sum of the exponents is always **5**. This always works: In every term in the expansion of $(a + b)^n$, the sum of the exponents is always $\boldsymbol{n}$.

3. The exponents on the a go down while the exponents on the b go up. Again, look at the form in the previous observation:

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Notice that the exponents on the a go down: 5, 4, 3, 2, 1, 0, while the exponents on the b go up: 0, 1, 2, 3, 4, 5.

EXAMPLE 1: **Apply the three observations above to the expansion of $\boldsymbol{(a + b)^{100}}$.**

Solution: First, there will be **101** terms in the expansion. Second, in any term the sum of the exponents will be **100**. For instance, the term with a^{40} in it will also have b^{60} in it. Third, the exponents on the a will go 100, 99, 98, . . ., 2, 1, 0 while the exponents on the b will go 0, 1, 2, . . . , 98, 99, 100. In short, using boxes for the unknown coefficients (the front numbers), the expansion of $(a + b)^{100}$ looks like this:

$$\Box a^{100} + \Box a^{99}b + \Box a^{98}b^2 + \cdots + \Box a^2b^{98} + \Box ab^{99} + \Box b^{100}$$

❑ PASCAL'S TRIANGLE

Now let's focus on the ***coefficients*** in the expansions we did on the first page (remember that the term a^4, for instance, is really $\mathbf{1}a^4$):

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1

Our goal is to extend this triangle of coefficients further by looking at the rows of the current triangle. Then we'll know the coefficients for even higher powers of $a + b$. Check out the **6** in the triangle; notice that it's the sum of the **3** and the **3** above it. Look at the first **10**; it's the sum of the **4** and the **6** above it. This pattern generates ***Pascal's Triangle***. Let's extend the triangle even further:

1	← **Row 0**
1 1	← **Row 1**
1 2 1	← **Row 2**
1 3 3 1	← **Row 3**
1 4 6 4 1	← **Row 4**
1 5 10 10 5 1	
1 6 15 20 15 6 1	
1 7 21 35 35 21 7 1	
1 8 28 56 70 56 28 8 1	← **Row 8**
1 9 36 84 126 126 84 36 9 1	← **Row 9**

Each entry of *PASCAL'S TRIANGLE* (starting in Row 2)
is the sum of the two entries above it.

Homework

8. Consider the expansion of $(a + b)^{23}$. How many terms are there in the expansion? In the term containing a^{15}, what is the exponent on the b?

9. There are 102 terms in the expansion of $(x + y)^n$. What is n?

10. Find the "10th row" of Pascal's Triangle.

❑ *AN EXAMPLE OF THE BINOMIAL THEOREM*

Let's use the three observations mentioned earlier, along with Pascal's Triangle, to deduce the expansion, without using algebra, of

$$(a + b)^7$$

1. There will be **8** terms in the expansion.

2. The sum of the exponents in each term will be **7**.

3. The exponents on the a will go from 7 down to 0, while those on the b will go from 0 up to 7.

   Also, the relevant row of Pascal's Triangle is

   1 7 21 35 35 21 7 1

   There are two ways we can know this. First, it's the only row in Pascal's Triangle with 8 entries in it, and we need 8 coefficients for the 8 terms of the expansion. Moreover, the second number in the row, the **7**, matches the exponent in the expression $(a + b)^7$.

Let's try it: $(a+b)^7$

$$= \underline{1}a^7b^0 + \underline{7}a^6b^1 + \underline{21}a^5b^2 + \underline{35}a^4b^3 + \underline{35}a^3b^4 + \underline{21}a^2b^5 + \underline{7}a^1b^6 + \underline{1}a^0b^7$$

The underlined coefficients come from Row 7 of Pascal's Triangle.

$$= \boldsymbol{a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7}$$

Is the Binomial Theorem worth knowing? Try expanding $(a+b)^7$ without it.

Homework

11. Find the first two terms and the last two terms of $(a+b)^{100}$.

12. Even though we know how to expand $(a+b)^2$ from basic algebra, show that the Binomial Theorem produces the same result.

13. Expand $(m+z)^8$.

14. Expand $(x+y)^{10}$.

Review Problems

15. How many terms are there in the expansion of $(a+b)^{200}$?

16. What is the sum of the exponents for any term in the expansion of $(a+b)^{123}$?

17. Consider the expansion of $(a + b)^{250}$.

a. How many terms are there in the expansion?
b. In the term containing a^{100}, what is the exponent on the b?
c. What is the first term of the expansion?
d. What is the second term of the expansion?
e. What is the second-to-last term of the expansion?
f. What is the last term of the expansion?

18. Expand $(u + w)^9$.

Solutions

1. They're boring, but do them!

2. a. 1 b. 2 c. 3 d. 4 e. 5 f. 6

3. a. 101 b. $n + 1$ **4.** a. $3 + 2 = 5$ b. 5

5. a. 4 b. 4 **6.** n

7. $k = 77$ **8.** 24; 8 **9.** $n = 101$

10. 1 10 45 120 210 252 210 120 45 10 1

11. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

12. To expand $(a + b)^2$ by the Binomial Theorem, we know that the relevant row of Pascal's Triangle is "**1 2 1**". We also know that the exponents on the a go from 2 down to 0, while the exponents on the b go from 0 up to 2. Putting it all together:

$$(a+b)^2 = \mathbf{1}a^2b^0 + \mathbf{2}a^1b^1 + \mathbf{1}a^0b^2 = a^2 + 2ab + b^2$$, as it should be.

13. $m^8 + 8m^7z + 28m^6z^2 + 56m^5z^3 + 70m^4z^4 + 56m^3z^5 + 28m^2z^6 + 8mz^7 + z^8$

14. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6$
$+ 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

15. 201 **16**. 123

17. a. 251 b. 150 c. a^{250} d. $250a^{249}b$
e. $250ab^{249}$ f. b^{250}

18. $u^9 + 9u^8w + 36u^7w^2 + 84u^6w^3 + 126u^5w^4 + 126u^4w^5 + 84u^3w^6$
$+ 36u^2w^7 + 9uw^8 + w^9$

"Cherish your visions and your dreams, as they are the children of your soul, the blueprints of your ultimate achievements."

– Napoleon Hill